Nonlinear Dynamic Effects of Adaptive Filters in Narrowband Interference-Dominated Environments

A. A. (Louis) Beex and Takeshi Ikuma

Abstract. When applying certain adaptive algorithms, such as the popular LMS and NLMS algorithms, in adaptive noise canceling or adaptive equalization scenarios where a strong narrowband interference component is present, these algorithms exhibit nonlinear dynamic behaviors. The latter is expressed in filter weights – that are generally expected to converge to constants – exhibiting a dynamic component. Furthermore, the performance of these adaptive filters with dynamic weight behavior can exceed the performance of any filter of the same structure in which those weights are fixed. In adaptive noise canceling applications, the dynamic component of the weights can be unmistakably large. In adaptive equalization scenarios, however, the dynamic aspect of the weight behavior can easily be mistaken for low level noise. These various findings will be illustrated.

1. Introduction

Adaptive filters achieving better mean-square error (MSE) performance than the fixed Wiener filter of corresponding structure has been observed in adaptive noise canceling, adaptive equalization, and adaptive prediction [1-6]. In all these cases there is a narrowband signal component present and – characteristically – the improved performance is associated with the use of large step-sizes in adaptation. The latter is suggestive of dynamic weight behavior as it is coupled to tracking.

In the adaptive noise canceling application, with a different center frequency of the narrowband component in the primary channel than the center frequency of the

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A. A. (Louis) Beex

Department of Electrical & Computer Engineering – Wireless@VT – DSPRL, Virginia Tech, Blacksburg, VA 24061-0111, e-mail: beex@vt.edu

Takeshi Ikuma

Department of Electrical & Computer Engineering, Louisiana State University, Baton Rouge 70803-5901, e-mail: tikuma@lsu.edu

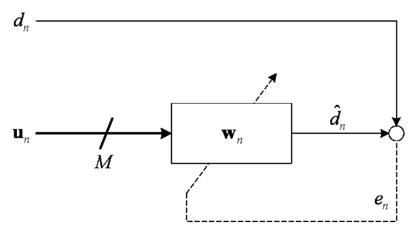

Fig. 1. General adaptive filter formulation.

narrowband component in the reference channel, a pronounced dynamic behavior in the weights of the adaptive filter may be observed. A hypothesized model for the origin of this behavior is based on the adaptive filter algorithm attempting to instantaneously track the underlying manifold of time-varying equivalents of a two-channel Wiener filter [6]. For an algorithm that entails more time-averaging the performance improvement disappears [7].

In the adaptive equalization application, where the equalizer is used to mitigate narrowband interference, the MSE performance can again be better than that of the Wiener filter of corresponding structure. However, in this application the dynamic weight behavior is of a much more subtle character, to the point that it can easily be thought to be a small random variation. It was recently shown that one aspect of the dynamic weight behavior is a pronounced change in the mean of the weight vector, away from the fixed Wiener weight vector [8, 9]. Again this effect happens in the presence of narrowband interference and when large step-sizes are used in adaptation.

In this paper we examine the dynamic behavior of the weights in the adaptive equalization case in more detail.

2. The LMS Algorithm

The LMS algorithm takes two input signals, a reference vector input $\mathbf{u}_n \in \mathbb{C}^M$ and a scalar primary input $d_n \in \mathbb{C}$ as shown in Fig. 1. The adapted weight vector $\mathbf{w}_n \in \mathbb{C}^M$ is updated by

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \mu \mathbf{u}_n e_n^* \tag{1}$$

where μ is the step-size parameter, * denotes the complex conjugate operation, and the a priori error signal is computed from

$$e_n = d_n - \mathbf{w}_n^H \mathbf{u}_n \tag{2}$$

with ^{H} denoting the Hermitian (conjugate) transpose operation.

While the LMS algorithm is popular, due to its simplicity of operation and numerical robustness, it is fundamentally a nonlinear algorithm as its coefficients depend on input values in a quadratic fashion. As readily seen from (1), when the step-size is large there is an instantaneous change in the weights that depends on the current input vector only; for smaller and smaller step-sizes the nonlinear aspects of the algorithm vanish, for they arise from the weight vector update term.

2.1. Traditional Statistical LMS Theory

The LMS algorithm is usually analyzed according to the following decomposition of its weights:

$$\mathbf{w}_n = \mathbf{w}_o + \mathbf{v}_n \tag{3}$$

where the corresponding Wiener solution is given by

$$\mathbf{w}_o \triangleq E\{\mathbf{u}_n \mathbf{u}_n^H\}^{-1} E\{\mathbf{u}_n d_n^*\} \tag{4}$$

and $\mathbf{v}_n$ is the deviation or misadjustment caused by the adaptation. The underlying assumption here is that the adaptation tries to identify $\mathbf{w}_o$ (assuming the input signals are wide-sense stationary) but the adapting weights $\mathbf{w}_n$ are subject to an unwanted deviation term $\mathbf{v}_n$ due to the adaptation taking place on the basis of data instead of on the basis of statistical averages.

One of the most popular measures, and one we utilize here, to assess the performance of the adaptive filter is the mean square error (MSE), $J \triangleq E\{|\, e_n \,|^2\}$. This performance metric is often viewed according to the weight decomposition in (3). The performance of the adaptive filter is generally believed to be bounded below by the mean square error (MSE) of the corresponding Wiener filter:

$$\begin{aligned} J_W &\triangleq E\{|\, d_n - \mathbf{w}_o^H \mathbf{u}_n \,|^2\} \\ &= \sigma_d^2 + \mathbf{w}_o^H E\{\mathbf{u}_n d_n^*\} \end{aligned} \tag{5}$$

Because it is the best possible mean square error attainable by a fixed filter, J_W in (5) is often referred to as the minimum mean square error. The error due to the deviations in the adapting weights is referred to as the excess mean square error.

The excess mean square error, both to characterize the transient and the steady-state behaviors of the LMS filter, has been studied usually for small step-sizes, where the dynamic behavior, if any, is largely suppressed. One such estimate for the MSE of the LMS algorithm is given by

$$J_{LMS,\infty} \approx J_W + \frac{\mu J_W}{2} \sum_{k=1}^{M} \lambda_k \quad \text{for } \mu \text{ small} \tag{6}$$

where $\{\lambda_k,\ k = 1, 2, \ldots, M\}$ are the eigenvalues of the input correlation matrix $E\{\mathbf{u}_n \mathbf{u}_n^H\}$ [10].

2.2. Dynamic weight behavior: departure from statistical theory

The above classical theory largely depends on Gaussian stochastic processes at the inputs. The behavior of the LMS algorithm can be substantially different when one or both of the input signals contain or consist of narrowband spectral components.

When adaptive noise canceling is used with sinusoidal interference the dynamic behavior of the adapting weights can be strongly periodic [11]. When the interference consists of narrowband autoregressive processes the adaptive filter weights typically exhibit semi-periodic dynamic behavior [6].

In the adaptive prediction and adaptive equalization applications, when narrowband components are present, performance can again be better than that of the Wiener filter with the same structure as the adaptive filter but with its coefficients fixed based on statistical quantities. This behavior is prominent at step-sizes that are large to very large compared to what is commonly used. While the non-Wiener or nonlinear performance improvement is observed in these latter two applications, the behavior of the adaptive filter weights is not prominently dynamic (as in the adaptive noise canceling case). An indication of beneficial dynamic weight behavior results from freezing the adaptive filter weights at a locally obtained time-average value, in which case any mean-square error performance improvement is turned into a mean-square error performance loss; if one wants fixed weights the Wiener weights produce the best performance in mean-square error sense.

We next analyze the subtle dynamic weight behavior in the adaptive equalizer application.

3. Adaptive Equalizer with Narrowband Interference

An application in which the dynamic behavior of the LMS algorithm becomes beneficial is when transversal adaptive equalizers are operating in an environment with strong additive narrowband interference. In digital communication, adaptive equalization is often utilized to mitigate the effect of a time-varying or unknown channel that causes intersymbol interference. Equalizers can also be used to alleviate other forms of interference such as additive narrowband interference. The latter function of an adaptive equalizer is our focus. When an adaptive equalizer is subject to strong additive narrowband interference, the LMS adaptation can

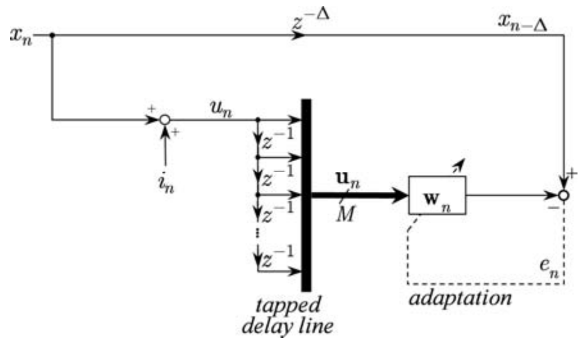

Fig. 2. Adaptive equalizer problem with narrowband interference.

provide much better performance than the optimal time-invariant Wiener equalizer both in mean-square-error (MSE) sense and in bit-error-rate sense [3, 4, 12].

The simplified adaptive equalizer problem with narrowband interference is shown in Fig. 2; the system is stripped down to the minimal set of components that cause the non-Wiener effect. First, the input u_n to the equalizer is merely the sum of the communication signal x_n and the interference i_n. In other words, the equalizer is purely responsible to mitigate the narrowband interference and not called upon to correct channel imperfections. Also, the receiver noise is assumed small enough to be negligible. Secondly, we assume that the entire transmitted symbol sequence is known a priori so that e_n is computed with the delayed version of the transmitted signal, $x_{n-\Delta}$. Typically, the adaptation error e_n is computed with the decision made by the receiver (decision-directed mode) or with the known training sequence (training mode). Our assumption is essentially that the equalizer is perpetually operating in either the training mode or in an error-free low noise environment (where the correct decision is made all the time). Lastly, we restrict ourselves here to considering complex sinusoidal interference as the narrowband interference, i.e., $i_n = \sigma_i \exp(j\omega_i n)$ with power σ_i^2 and frequency ω_i rad. While the dynamic weight behavior can be observed under stochastic narrowband interference, the current analysis does not directly extend to such stochastic interference.

Fig. 3 illustrates the MSE performance of the LMS adaptive equalizer as a function of the LMS step-size based on numerical simulation[1]. The LMS MSE, which is shown relative to the Wiener equalizer MSE, depends on only two parameters: the number of filter taps *M* and the interference-to-signal (ISR) ratio σ_i^2 / σ_x^2 with the communication signal power σ_x^2. The step-size is normalized with respect to the number of equalizer taps *M* and the received signal power $\sigma_u^2 = \sigma_x^2 + \sigma_i^2$. The illustrated MSE behavior of the adaptive equalizer is

[1] Three system parameters are fixed in all the simulations: $\Delta = 0$, $\sigma_x^2 = 1$, and $\omega_i = 0.2\pi$ rad/sample. Also, the simulations use a random sequence of 8-QAM symbols as the communication signal in all the simulations.

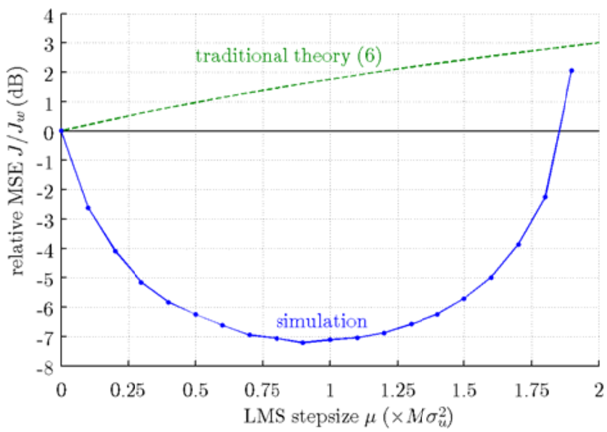

Fig. 3. MSE performance (Monte Carlo) as a function of step-size: M = 20, ISR = 20 dB.

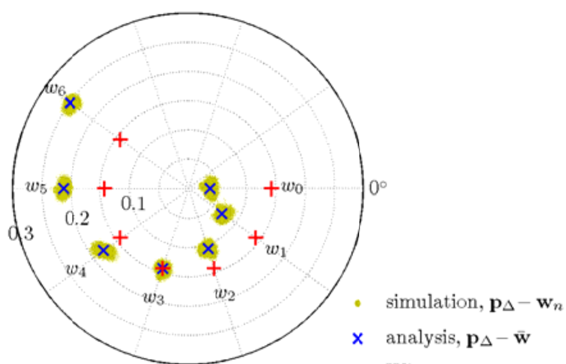

Fig. 4. Steady-state weight behavior of LMS equalizer (M = 7, $\mu = (M\sigma_u^2)^{-1}$, ISR = 20 dB).

completely unexpected from the traditional analysis result in (6). The adaptive filter is clearly outperforming the corresponding fixed Wiener equalizer as indicated by the negative relative MSE.

In addition to the MSE-sense performance improvement, the dynamic weight behavior of the LMS equalizer interacting with narrowband interference causes an interesting side effect: a prominent shift in the time-averaged weights. This behavior is different from that in the adaptive noise cancellation application, where the prominent feature is dynamic pseudo-periodic weight behavior close to the Wiener solution (which was all zeros in the purely sinusoidal case). The steady-state behavior of the LMS equalizer weights is illustrated in Fig. 4. The deviation of the LMS weights from the Wiener solution is prominently visible. The variation of the weights about their mean value is indicated by the little "clouds" of 10,000 consecutive weight values, i.e. the changes in the weights are small relative to their mean value, which deviates substantially from the Wiener weight values.

The spiral-shaped mean of the LMS equalizer weights has been analyzed for the strong sinusoidal interference case and is found to be [8]

$$\bar{\mathbf{w}} = \mathbf{p}_\Delta - \frac{\sigma_i^2}{\lambda_{\max}}\left(\mathbf{I} - \mu\Xi\Theta\right)^{-1}\mathbf{e} \tag{7}$$

where

$$\mathbf{p}_\Delta \triangleq [\underbrace{0 \quad \cdots \quad 0}_{\Delta} \quad 1 \quad 0 \quad \cdots \quad 0]^T, \tag{8}$$

$$\lambda_{\max} = \sigma_x^2 + M\sigma_i^2, \tag{9}$$

$$\mathbf{e} = e^{j\omega_i\Delta}\left[1 \quad e^{-j\omega_i} \quad \cdots \quad e^{-j\omega_i(M-1)}\right]^T, \tag{10}$$

$$\Xi \triangleq \frac{1}{\sigma_x^2}\left(\mathbf{I} - \frac{\sigma_i^2}{\lambda_{\max}}\mathbf{1}\mathbf{1}^H\right), \tag{11}$$

and

$$\Theta \triangleq \frac{\sigma_i^2 M}{\sigma_x^2}\sum_{p=1}^{M-1}\mathbf{Z}^p\left(1 - \mu\lambda_{\max}\right)^{p-1}. \tag{12}$$

In the above, $\mathbf{I}$ is the identity matrix, $\mathbf{1}$ is the vector of all ones, and $\mathbf{Z}$ is the lower triangular shift matrix (ones on the subdiagonal and zeros elsewhere).

The mean offset itself does not explain the performance improvement of the adaptive equalizer over the Wiener equalizer. If anything, it suggests that the LMS equalizer should underperform because it is a suboptimal solution. Instead, the time-varying portion of the weights, which is smaller than the magnitude of the mean portions, must be responsible for the MSE performance. Inspired by the time-varying subspace analysis for the adaptive noise canceller [13, 14], we propose the following dynamic weight model for LMS used in our problem:

$$\mathbf{w}_n = \bar{\mathbf{w}} + \mathbf{e}\alpha_n + \boldsymbol{\beta}_n \tag{13}$$

The LMS weights consist of the fixed portion $\bar{\mathbf{w}}$ and the time-varying portion $\mathbf{e}\alpha_n + \boldsymbol{\beta}_n$. Of the two time-varying components, $\mathbf{e}\alpha_n$ represents the dynamic behavior while the vector process $\boldsymbol{\beta}_n$ is the weight misadjustment. The dynamic components of the weights are locked to each other by $\mathbf{e}$, and their collective

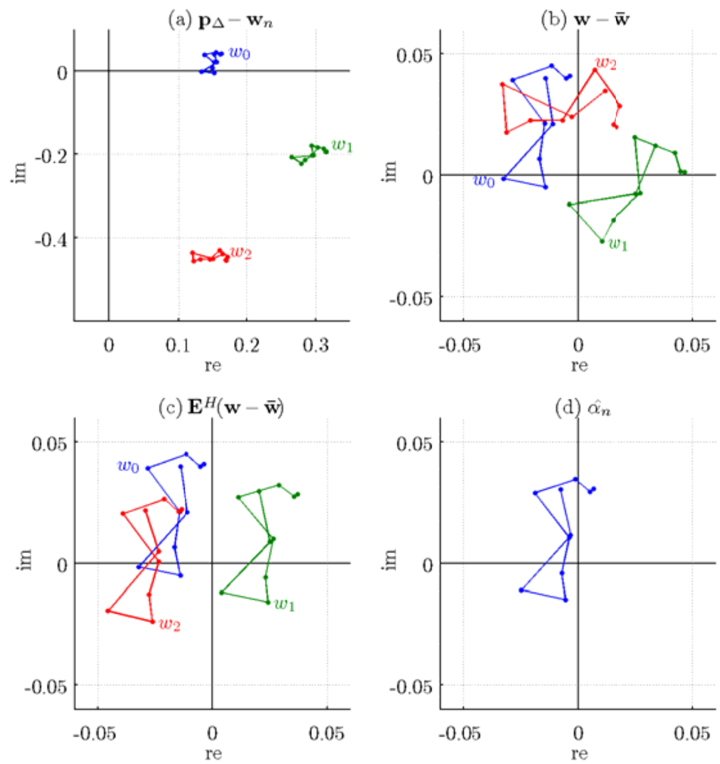

Fig. 5. Dynamic weight behavior deconstruction for the LMS equalizer ($M = 3$, $\mu = (M\sigma_u^2)^{-1}$, ISR 20 dB): (a) 10 samples of the LMS weight vector components, (b) after removal of the mean, (c) after undoing the rotation, and (d) the estimated α_n.

motion α_n is a lowpass stochastic process. The misadjustment weights $\boldsymbol{\beta}_n$ are assumed to be zero-mean.

The model in (13) is verified in this paper by estimating α_n with

$$\hat{\alpha}_n = \frac{1}{M}\sum_{m=0}^{M-1}\left(w_{m,n} - \overline{w}_m\right)e^{j\omega_i(m-\Delta)} \tag{14}$$

where $w_{m,n}$ and $\overline{w}_m$ indicate the m-th element of the weight vectors $\mathbf{w}_n$ and $\overline{\mathbf{w}}$, respectively. Subsequently, the weights are estimated by

$$\hat{\mathbf{w}}_n = \overline{\mathbf{w}} + \mathbf{e}\hat{\alpha}_n \tag{15}$$

The MSE can be computed for the estimated weights in (15) with the same input and desired signals as $\mathbf{w}_n$. Fig. 5 illustrates how the LMS weights relate to α_n.

Fig. 5(b) and Fig. 5(c) clearly illustrate the rotated nature of the time varying weights and their similarities in the signals. The observed offsets among the rotated time-varying weights in Fig. 5(c) relate to the $\boldsymbol{\beta}_n$ process and are not constant over time; instead, the offsets represent low-pass stochastic processes and thus change slowly over time.

Furthermore, as functions of ISR, Fig. 6 compares the experimentally evaluated MSE (relative with respect to the Wiener MSE) of the LMS error e_n with the MSE estimated using the mean weights only, and with the MSE estimated using the time-varying weight model in (15). The latter error signals are computed as follows.

$$\bar{e}_n = d_n - \bar{\mathbf{w}}^H \mathbf{u}_n \tag{16}$$

and

$$\hat{e}_n = d_n - \hat{\mathbf{w}}_n^H \mathbf{u}_n \,. \tag{17}$$

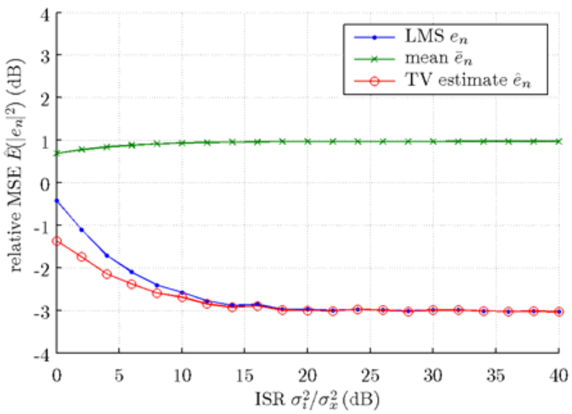

Fig. 6. Estimated MSEs of the LMS equalizer – relative to Wiener MSE – using the proposed model ($M = 7$, $\mu = (M\sigma_u^{\,2})^{-1}$).

A 7-tap LMS equalizer with step-size $\mu = \sigma_u^{-2}$ is used in these Monte Carlo simulations. There are three notables in this result in support of the validity of our model. First, the proposed $\mathbf{e}\alpha_n$ dynamic model indeed characterizes the non-Wiener performance gain. In addition, the MSE of the estimated model is approaching the simulation from below, supporting the proposed role of $\boldsymbol{\beta}_n$ as the misadjustment component that causes excess error. Lastly, the difference between e_n and $\hat{e}_n$ asymptotically disappears as the interference becomes stronger, implying that $\boldsymbol{\beta}_n$ vanishes as the interference increasingly dominates.

Next, the statistical properties of the $\hat{\alpha}_n$ process are evaluated. This simulation is conducted in a 20-dB ISR environment and for step-size $\mu = (M\sigma_u^2)^{-1}$. The autocorrelation functions and power spectra of $\hat{\alpha}_n$ are examined for different equalizer tap-lengths: M = 3, 7, and 20. The autocorrelation functions are shown in Fig. 7, and the power spectrum estimates are given in Fig. 8. The autocorrelation functions clearly depend on M: $\hat{\alpha}_n$ looks to be predominantly an MA(M) (M-th order moving average) process. We observe these processes to be low-pass in nature, with a bandwidth relating to the M parameter.

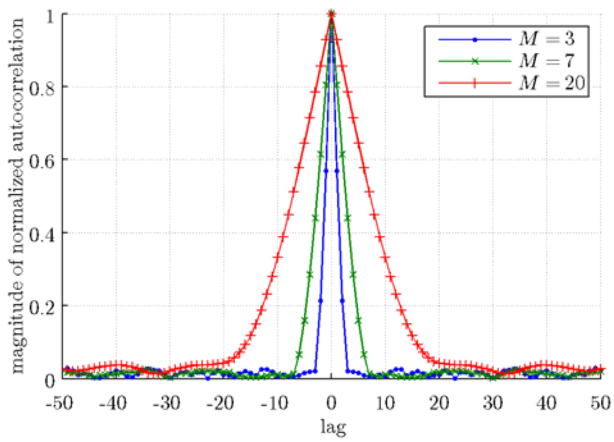

Fig. 7. Normalized autocorrelation functions of the estimated $\hat{\alpha}_n$ (ISR = 20 dB, $\mu = (M\sigma_u^2)^{-1}$).

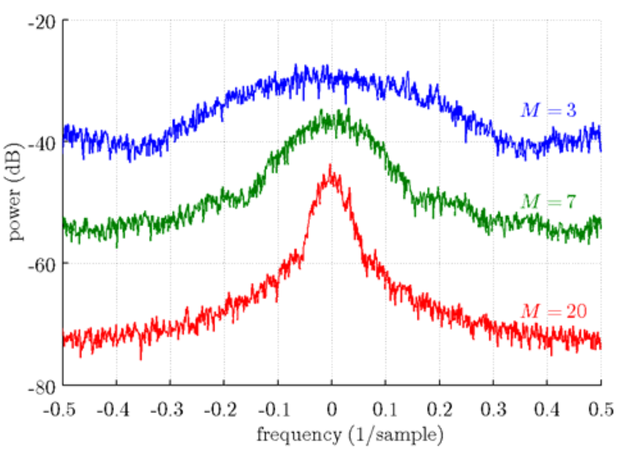

Fig. 8. Power spectrum estimates of the estimated $\hat{\alpha}_n$ (ISR 20 dB, $\mu = (M\sigma_u^2)^{-1}$).

The last illustration in Fig. 9 shows estimates for the variance of $\hat{\alpha}_n$ in (14) and for $\hat{\beta}_{0,n}$, the 0-element of $\hat{\boldsymbol{\beta}}_n \triangleq \mathbf{w}_n - \hat{\mathbf{w}}_n$, for different tap lengths and ISRs. The other elements of $\hat{\boldsymbol{\beta}}_n$ behave similarly to $\hat{\beta}_{0,n}$. We observe that both time-varying components have similar variance for ISR below 30 dB; for higher ISR the variance of $\hat{\beta}_{0,n}$ starts to drop off. In Fig. 6 we saw that beyond 15 dB ISR there was no visible effect $\hat{\boldsymbol{\beta}}_n$ on the LMS estimation error. We verified that this is a result of $\hat{\boldsymbol{\beta}}_n^H \mathbf{u}_n$ vanishing, i.e. $\hat{\boldsymbol{\beta}}_n$ becomes orthogonal to $\mathbf{u}_n$ while $\mathbf{e}\hat{\alpha}_n$ does not.

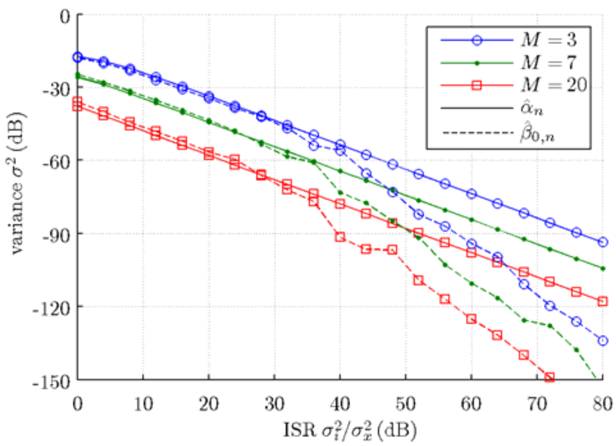

Fig. 9. Variance estimates of the two time-varying components of the proposed LMS weight model ($\mu = (M\sigma_u^2)^{-1}$).

The above results point to the future work refinement of the present model, in which the α_n process is modeled explicitly as MA(*M*). The latter will allow us to add $\boldsymbol{\beta}_n$ explicitly to the model as the misadjustment process so that it can be estimated simultaneously with the MA(*M*) parameters. In addition, we plan to connect the time-varying weight behavior directly to the input signals.

4. Conclusions

We have presented the beneficial time-varying weight behavior of the popular LMS adaptive algorithm. The LMS algorithm, when applied in the presence of narrowband signals, such as the narrowband interference discussed in this paper, can outperform the Wiener mean-square error statistical framework that the algorithm is derived from. We have proposed a new model for the dynamic behavior of the LMS weights of the adaptive equalizer mitigating strong narrowband interference. The model has been shown experimentally to converge asymptotically towards the actual LMS behavior as the input becomes more interference dominated.

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