

A TIME-VARYING PRONY METHOD FOR INSTANTANEOUS FREQUENCY ESTIMATION AT LOW SNR

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ABSTRACT

Most available instantaneous frequency (IF) estimation methods for frequency modulated signals in white noise deteriorate dramatically when SNR falls below some threshold. We present a time-varying Prony method for IF estimation from low SNR data. It is an extension of a frequency estimation method for stationary processes which was shown to be less complicated than, yet close in performance to, the best available approaches based on singular value decomposition or the principle of maximum likelihood. First, the time-varying autoregressive order is set higher than that needed for pure signal components so that the extra poles capture part of the noise. Then we choose “signal poles” based on a subset selection procedure. The performance improvement at low SNR over the TVAR method without subset selection is evidenced through simulation experiments.

1. INTRODUCTION

Instantaneous frequency (IF) estimation for frequency modulated (FM) signals in white noise is a classical problem. The available methods range from the classical counting of zero-crossings and phase-locked-loop (PLL) to the recently developed time-frequency distribution (TFD) based methods [1]. In high signal to noise ratio (SNR) environments many methods yield accurate IF estimates, and even reach the Cramer-Rao bound in some situations, whereas most of these methods deteriorate dramatically when SNR falls below some threshold [1].

Time-varying autoregressive (TVAR) model based IF estimation had been considered poor since being proposed in 1984 by Sharman and Friedlander [9,1]. Shan and Beex [8] have recently shown that it is a fairly good method, and especially advantageous for those practical cases where

short data records are used and/or a linear IF law can not be assumed *a priori*. This is further evidenced by the superior system performance provided by the TVAR based method when applied to interference cancellation in spread spectrum communications [7].

On the other hand, our results [8] also revealed that, similar to other IF estimation methods [1], the reciprocal of MSE of the TVAR based IF estimation falls faster after the SNR falls below some value. In this paper, we focus on improving this method for low SNR environments.

In stationary cases, sinusoids in white noise can be approximately modeled as an AR process, and the frequencies can be estimated from the model. This AR model based frequency estimation method is referred to as the Prony method [5, 6]. At low SNR, the eigen-structure of the data correlation matrix can be exploited to improve frequency estimation precision significantly [5]. An alternative, less complicated, method to solve this problem was proposed by Kumaresan, Tufts, and Scharf [6]. The performance of this alternative approach was shown to be [6] “close to that of the best available, more complicated, approaches which are based on maximum likelihood or on the use of eigenvector or singular value decompositions.”

We extend this less complicated low-SNR Prony method to time-varying cases to improve the TVAR based IF estimation at low SNR. First, the autoregressive order is set higher than that needed for pure signal components so that the extra poles capture part of the noise. Then we choose “signal poles” based on a subset selection procedure. Simulations show significant improvement of the IF estimation, especially when SNR is low.

In the rest of the paper, the TVAR model is reviewed in Section 2, a time-varying Prony method for IF estimation in low SNR is introduced in Section 3, and some simulation results are presented in Section 4.

2. TVAR MODEL REVIEW

For ease of reference, we briefly review the time-varying AR model in this section. For details of the TVAR model, readers are referred to Grenier [2] and Hall, Oppenheim and Willsky [3]. Studies of TVAR model based IF estimation in high to moderate SNR conditions were presented by Sharman and Friedlander [9] originally, as well as by Shan and Beex [8] recently. Application of this method to the problem of FM-type interference cancellation in direct-sequence spread spectrum communications was also studied [7].

A discrete-time time-varying autoregressive (TVAR) process $x(t)$ of order p is expressed as

$$x(t) = -\sum_{i=1}^p a_i(t)x(t-i) + e(t) \quad (1)$$

where $e(t)$ is a stationary white noise process with zero mean and variance σ^2 , and the TVAR coefficients $\{a_i(t), i=1,2,\dots,p\}$ are modeled by linear combinations of a set of basis time functions $\{u_k(t), k=0,1,\dots,q\}$:

$$a_i(t) = \sum_{k=0}^q a_{i,k}u_k(t) \quad (2)$$

where $\{u_k(t), k=0,1,\dots,q\}$ can be any appropriate set of basis functions. If $\{u_k(t)\}$ are chosen as powers of time, then $\{a_i(t)\}$ are polynomial functions of time t . If $u_k(t)$ are trigonometric functions, then (2) is a finite order Fourier series expansion. In any case, the TVAR model is described completely by the set of parameters $\{a_{i,k}, i=1,2,\dots,p; k=0,1,\dots,q; \sigma^2\}$.

The estimation of $\{a_{i,k}\}$ aims at minimizing the total squared prediction error in predicting the sequence $x(t)$:

$$E = \sum_t \left| x(t) + \sum_{i=1}^p \sum_{k=0}^q a_{i,k}u_k(t)x(t-i) \right|^2 \quad (3)$$

If we define the generalized covariance function as

$$c_{kl}(i,j) = \frac{1}{N-p} \sum_{t=p}^{N-1} u_k(t)u_l(t)x(t-i)x(t-j) \quad (4)$$

then the solution $\{a_{i,k}, i=1,2,\dots,p, k=1,2,\dots,q\}$ that minimizes (3) can be solved for from the generalized covariance equations:

$$\sum_{i=1}^p \sum_{k=0}^q a_{i,k}c_{kl}(i,j) = -c_{0l}(0,j),$$

$$1 \leq j \leq p, 0 \leq l \leq q \quad (5)$$

This is a system of $p(q+1)$ linear equations.

The basis function set and orders p and q should be selected using *a priori* knowledge of the signal. In this paper, we use polynomials in time to approximate the time-varying coefficients.

3. METHOD

For a signal consisting of M FM components in white noise with low to moderate SNR, we model the signal with a TVAR model, with order $p > M$ for complex exponential FM components and $p > 2M$ for real signals. M is assumed known here. For the sake of presentation, in the rest of the paper we assume to be using complex data, consisting of an analytic signal in complex noise.

The time-varying transfer function corresponding to the TVAR model can be expressed as

$$H(z,t) = \frac{1}{1 + \sum_{i=1}^p a_i(t)z^{-i}} \quad (6)$$

By rooting the denominator polynomial formed by the TVAR coefficient estimates at each time instant t , we can obtain the p poles as functions of time: $p_i(t), i=1,2,\dots,p$. Among the p pole trajectories, there are M signal pole trajectories and $(p-M)$ noise pole trajectories. The trajectories associated with the FM components tend to be along the unit circle while the poles capturing the noise move wildly, mostly inside, but occasionally outside the unit circle (especially near the ends of a data record).

The instantaneous angles of the pole trajectories associated with the FM components can be used as estimates of the instantaneous frequencies $f_i(t)$. The key problem here is to choose the best subset of the poles as the M signal poles at each instance t .

Our objective is to find the best subset of size M out of the p poles at each moment so that the M trajectories formed from the subsets will provide the least total squared prediction error over the entire data record. For data size

N , we have $\binom{p}{M}^{N-p}$ potential combinations to choose from. Although we may use, for example, the Viterbi algorithm to solve this combinational optimization

problem, it is still desirable to have a less complicated way to find a subset of poles to improve the IF estimation for low SNR environments.

To meet this demand, we suggest a subset selection method as follows. First, we use the pole radius distribution at each time index. If there are M poles close to the unite circle, while all others are far away from it, the decision is made to take those M poles as the subset without further calculation. Second, if the desired pole radius distribution does not hold, we evoke a secondary criterion, which is the instantaneous squared prediction error. At time t , the autoregressive coefficients associated with a pole subset, denoted as ψ , of size M , are determined by

$$\sum_{i=0}^M a_i^\psi(t) z^{-i} = \prod_{m \in \psi} [1 - p_m(t) z^{-1}] \quad (7)$$

Then the squared prediction error at time t corresponding to the subset ψ is

$$\epsilon^\psi(t) = \left| x(t) - \sum_{i=1}^M a_i^\psi(t) x(t-i) \right|^2 \quad (8)$$

We choose the subset that gives the smallest $\epsilon^\psi(t)$.

At each time, there are $\binom{p}{M}$ possible subsets of size M .

For cases of large M and p , the procedure of Hocking and Leslie [4] can be used to simplify the subset selection, as suggested by Kumaresan, Tufts, and Scharf for the stationary case [6]. In this procedure, the significance of each pole is evaluated in terms of the prediction error increment after removing the pole from the entire set of p poles. All poles are sorted according to the significance measure, and from this ordering, the best subset can be selected without exhausting all the possible combinations. It might be possible to combine the criteria in the above two steps into a single criterion by taking a weighted sum of the squared distances of the poles to the unit circle and the squared instantaneous prediction error as the composite objective. However, determination of the relative weighting or normalization of each contributing error criterion remains to be investigated.

4. SIMULATION RESULTS

To observe the performance of the proposed IF estimator for noisy data, we compare it with two other methods: the TVAR based IF estimation for moderate to high SNR where $p=M$ [8], and the WVD peak based IF estimator [1].

4.1 A Nonlinear FM Case

We use 32 samples at unit sampling rate of a non-linear FM signal, which chirps non-linearly from frequency 0.05 to about 0.4 Hz (normalized to sampling rate) according to $f(t) = 0.05 + 0.004t^2, t = 0, 1, \dots, 31$ and is corrupted by additive white Gaussian noise. The signal is generated using the IF with unit amplitude and a random initial phase. Zero-padding to 256 points is applied in the WVD method to reduce frequency quantization error. For the proposed method, $M=1, p=2$, and $q=3$ are used. We run 200 simulations and the IF estimation errors in the center half of the data record (t from 16 to 23) are evaluated. The result is shown in Figure 1 in terms of the reciprocal of the mean squared error (MSE) of IF estimation, $1/\text{MSE}$, in dB versus SNR. Noticeable performance improvement, over the moderate to high SNR TVAR method, is achieved for SNR below 10dB. The WVD method performs poorly mainly due to its biased IF estimation in this case [8].

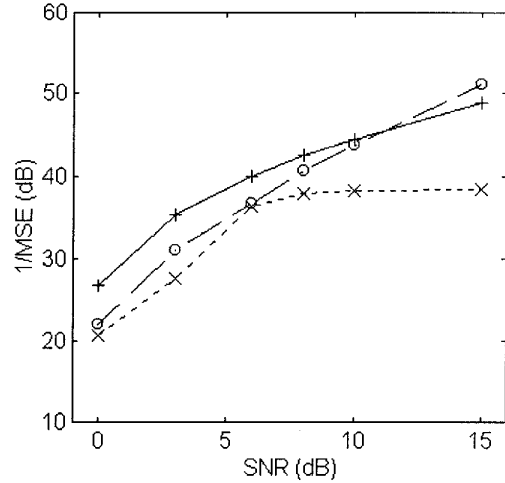


Figure 1. $1/\text{MSE}$ vs. SNR for IF estimation near the time center of the data record for a nonlinear FM signal: proposed method (solid with +), TVAR for high SNR (dashed with o), and WVD method (dots with X).

4.2 A Linear FM Case

We now use 32 samples at unit sampling rate of a linear FM signal, which chirps from frequency 0.1 to about 0.4 Hz, according to $f(t) = 0.1 + 0.01t, t = 0, 1, \dots, 31$. In all other respects the experiment is the same as in the nonlinear FM case in Section 4.1. The result is shown in Figure 2. Again, noticeable performance improvement is achieved for SNR below 10dB. The WVD method is potentially optimal for this case due to pure linearity of the

FM law [1,8]. The optimality of the WVD method seems to materialize when SNR is above 6dB. However, at lower SNR (below 3dB for this example), the proposed method provides the best estimation of the three methods that were compared here.

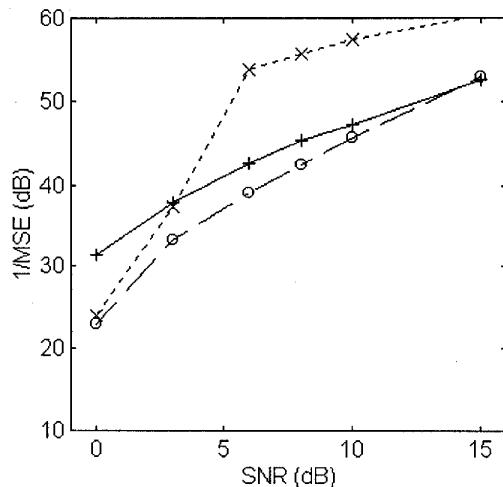


Figure 2. 1/MSE vs. SNR for IF estimation near the time center of the data record for a linear FM signal: proposed method (solid with +), TVAR for high SNR (dashed with o), and WVD method (dots with X).

5. CONCLUSION

We presented a time-varying Prony method for IF estimation from low SNR data. The performance improvement at low SNR over the TVAR method for high SNR (without subset selection) is evidenced through simulation experiments. The proposed method also outperforms the "optimal" WVD method, when it loses optimality for low SNR scenarios.

6. REFERENCES

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